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Symmetry breaking and tensor operator techniques

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Abstract. We review symmetry breaking from the standpoint of the state labelling inherent in the selection of a particular little subgroup of the original symmetry group, by the symmetry breaking perturbation. In the case of non-degenerate multiplets, we show in a general tensor operator formalism how the expectation values of the perturbation yield algebraic relations of the Gell-Mann Okubo type, and give examples in SU(u), O(n) and Sp(n). In the case of degenerate multiplets, we extend these tensor operator techniques, and also give a prescription for calculating the transition matrix elements of the perturbation, illustrated by examples in SU(n), O(n) and Sp(n). These examples entail the evaluation of certain recoupling coefficients in SU(n), O(n) and Sp(n) which are given explicitly.

1. Introduction

Developments in particle physics in the last few years have led to renewed interest in the possibility of new quantum numbers and new symmetries. Recent experimental discoveries are thought to herald whole new generations of hadrons (even beyond the 'new' Ψ physics) and of leptons (Herb *et al* 1977, Perl *et al* 1975). This has been reflected in theoretical proposals incorporating on the one hand higher phenomenological 'flavour' symmetries and, on the other, higher unified gauge symmetries with increasing numbers of 'elementary' quarks and leptons. Moreover, groups other than unitary groups are being actively considered. For example, the model of Barnes *et al* (1978) was based on a phenomenological SO(8) symmetry, and extended to a non-relativistic Sp(16) spin symmetry. Indeed, in some versions of extended supergravity, only orthogonal symmetry groups arise (for a review see Wess 1977). There seem to be only weak restrictions on possible candidates for unified gauge groups (see for example Gell-Mann *et al* 1977).

In the light of this interest, and for other applications, it is felt timely and useful to take stock of methods of extracting whatever information is available from symmetry arguments alone, independently of dynamical considerations. The aim of this paper is to present a formalism for handling symmetry breaking in perturbation theory, which reproduces known results in the non-degenerate case, and, as we shall show, extends to the treatment of the case of degenerate multiplets.

We firstly review symmetry breaking, emphasising that the introduction into the Hamiltonian, H, of a perturbation, V, breaking the symmetry from a Group G to a group K entails a corresponding choice of state labelling in the multiplets of G (§ 2). In § 3, taking G to be one of the classical groups SU(n), O(n) or Sp(n), we introduce a general notation for their generators, and consider which irreducible tensor operators

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are available as candidates for symmetry terms, in the enveloping algebras. Perturbation theory requires the knowledge of expectation values $\langle i|V|i\rangle$, and transition amplitudes $\langle f|V|i\rangle$. For the non-degenerate case, we retrieve with our construction known results of the Gell-Mann Okubo type for the expectation values, and we illustrate with some examples in SU(n), O(n) and Sp(n) (§ 4). For the case of degenerate multiplets of G, the transition amplitudes are required even in lowest order, and in § 5 we extend the tensor operator formalism to handle this situation. The techniques developed are illustrated with examples in SU(n), O(n) and Sp(n). Some concluding remarks are made in § 6.

2. Symmetry breaking and state labelling

Consider a system with a Hamiltonian of the form

$$H = H(0) + \epsilon V$$

where, for $\epsilon = 0$, $H(0)$ is invariant under transformations by elements g of some group, G:

$$H(0) = U_g H(0) U_g^{-1}$$

but not invariant for $\epsilon \neq 0$. The cases $\epsilon = 0$ and $\epsilon \neq 0$ are called unbroken and broken symmetry respectively. In general, there will be some group K, for which V, and therefore H itself, are still invariant, for $\epsilon \neq 0$. We assume that K is a subgroup of G. K is called the little group of V, and the symmetry is said to be broken from G to K.

The physical picture is that, for $\epsilon = 0$, the eigenstates of the system will be arranged into a number of multiplets, within each of which all eigenstates have the same energy. These multiplets correspond to irreducible representations of G. When the breaking term is turned on, and $\epsilon \neq 0$, these multiplets of G will, in general, split into submultiplets, corresponding to irreducible representations of K.

If the breaking term is small, we can hope to perturb about the unbroken situation. In order to diagonalise V as well as H, it is necessary to choose the correctly adapted unperturbed (zeroth order) states. Namely, we must arrange states in representations of G according to which representations of K they belong. This amounts precisely to a choice of basis for labelling irreducible representations of G, given by a subgroup labelling chain which includes K. Specifically, let us denote irreducible representations of G by $\{\alpha\}$, and of K by $\{\kappa\}_X$ where X indicates that K may include some additive quantum numbers which remain good in the presence of the breaking. Then states are labelled as

$$\left| \begin{array}{l} \{\alpha\} \\ \Gamma \ldots \\ \{\kappa\}_X \\ \ldots \end{array} \right\rangle$$

Here $\Gamma \ldots$ stands for a set of additional labels, which may be additional subgroup labels in the chain $\mathrm{G} \supset \ldots \supset \mathrm{K}$, or they may include non-subgroup invariants, required to specify completely the basis, if the reduction $\mathrm{G} \supset \mathrm{K}$ is degenerate. Γ will be suppressed in the following discussion. Also, '. . .' stands for a set of labels which are required to specify states within the same K submultiplet; for our present purposes, we do not need to consider them explicitly. For more details of the history of work on symmetry breaking and state labelling ideas, see for example Butler 1975.

Although V is itself not invariant under G, there are very often physical grounds for its being decomposable into a sum of parts which at least have well-defined transformation properties under G. That is, in general it can be expressed as a sum of components of tensor operators, transforming under some irreducible representations of G†. For simplicity, we shall take V to have one irreducible part, belonging to an irreducible representation $\{\beta\}$ of G.

However, since K is the little group of V, it is clear that, in terms of our previous labelling,

$$V \sim \left| \begin{matrix} \{\beta\} \\ \{0\}_0 \end{matrix} \right\rangle \tag{1}$$

where '$\sim$' means 'transforms like', and $\{0\}_0$ represents a singlet of K.

To treat H in perturbation theory, we require matrix elements of V. Using (1) and the Wigner–Eckart theorem (see for example Butler 1975), these can be expressed as a product of a reduced matrix element (one for each $\{\alpha\}$, $\{\beta\}$ and $\{\gamma\}$) and a Clebsch–Gordan coefficient,

$$\left\langle \begin{matrix} \{\alpha\} \\ \{\kappa\}_X \\ \ldots \end{matrix} \middle| V \middle| \begin{matrix} \{\gamma\} \\ \{\mu\}_Z \\ \ldots \end{matrix} \right\rangle = \langle\{\alpha\}\|\{\beta\}\|\{\gamma\}\rangle \left\langle \begin{matrix} \{\alpha\} \\ \{\kappa\}_X \\ \ldots \end{matrix} \middle| \begin{matrix} \{\beta\} & \{\gamma\} \\ \{0\}_0\,; & \{\mu\}_Z \\ & \ldots \end{matrix} \right\rangle . \tag{2}$$

From this, it is clear that we must have $\{\kappa\} = \{\mu\}$ and $X = Z$ for a non-zero result. If the multiplets $\{\alpha\}, \{\gamma\}, \ldots$ of G are non-degenerate, then in lowest order we only require expectation values of V to calculate the spectrum (as was to be expected from our choice of basis for labelling states). In higher order, or in lowest order with degenerate multiplets of G, transition matrix elements of V, with $\{\gamma\} \neq \{\alpha\}$, are required.

This discussion shows that, under general assumptions about the form of the symmetry breaking, the problem is summarised by a certain number of reduced matrix elements (which can only be calculated in a more specific model), and by Clebsch–Gordan coefficients of the form

$$\left\langle \begin{matrix} \{\alpha\} & \{\beta\} & \{\gamma\} \\ \{\kappa\} & \{0\}_0; & \{\kappa\}_X \\ \ldots & & \ldots \end{matrix} \right\rangle . \tag{3}$$

We have made no attempt to pay careful attention to detail in the above discussion. In particular, the question of the number of reduced matrix elements involved is to be examined case by case; this may also depend on further assumptions (for example, that distinct irreducible parts of V are different components of the same irreducible tensor: see Feldman and Matthews 1978, and Okubo *et al* 1975). Rather, the aim of this paper is to present techniques for calculating the Clebsch–Gordan coefficients (3). More exactly, since the latter are obviously the same for all states of the K-submultiplet $\{\kappa\}_X$, our techniques are concerned with the calculation of precisely those singlet factors for $\{\alpha\}$ contained in $\{\beta\} \times \{\gamma\}$ of G, in which the submultiplet corresponding to $\{\beta\}$ is a singlet $\{0\}_0$.

In §§ 4 and 5 below, we shall treat the cases $\{\gamma\} = \{\alpha\}$ and $\{\gamma\} \neq \{\alpha\}$ respectively. Firstly, however, we consider tensor operators in more detail.

† The concept of tensor operator was developed by Wigner and Racah.

3. Tensor operators and the enveloping algebras of U(n), SU(n) O(n) and Sp(n)

Early work, on broken SU(3) symmetry (Gall-Mann 1962, Okubo 1962), and subsequent developments (see for example Pais 1966), centred mainly around the assumption that the symmetry breaking term should be a component of the adjoint representation of G (a 'vector' operator, in analogy with SU(2)). It was suggested (see for example Ginibre 1963, Lehrer-Ilamed and Goldberg 1963), finally proven for SU(n) (Okubo 1975), and extended to O(n) and Sp(n) (Rashid and Nwachuku 1976, Okubo 1977), that any such adjoint operator can be written as a polynomial in the generators of G, in an irreducible representation $\{\alpha\}$. For SU(n), owing to the existence of an nth order characteristic identity (Green 1971), this polynomial is in general of order $(n-1)$, or of lower order if $\{\alpha\}$ corresponds to a non-generic combination of fundamental highest weights; a similar statement obtains for the other simple Lie algebras (Ginibre 1963, Bracken and Green 1971, Okubo 1975). In practice, this means that for the symmetry breaking term, a component of the adjoint operator, the expectation values can be written as a linear combination of terms recognizable as invariants of the little group K (whose eigenvalues can in principle be written down in terms of the highest-weight labels of the submultiplet $\{\kappa\}_X$, and other labels higher in the labelling chain), together with unknown reduced matrix-elements, which can be fitted by comparison with the experimental masses, magnetic moments, and so on (examples will be given in the next section).

In the present work, we wish to generalise this procedure to include the possibility of calculating transition amplitudes. Moreover, we remove the previous restriction to the adjoint operators only (as was found necessary early on, see Pais 1966). In this more general situation, there is no longer any proof available that the tensor operator in question can be realised as a polynomial in the generators of G (an element of the enveloping algebra of G). Nonetheless, we shall continue to assume that it is useful to utilise the enveloping algebra for the construction of symmetry breaking terms. We take a pragmatic attitude to the problems of the sufficiency and completeness of this approach: for, in any given example, we can always determine separately the number of independent couplings involved, and ascertain whether or not in the algebraic technique the correct number of independent terms is present. For the examples which we take, this will indeed be the case.

There is one fundamental limitation to our general goal of constructing tensor operators in the enveloping algebra of G, which should be pointed out. It is that only certain irreducible representations are available. For, if we designate the generators of G by $X_\rho, X_\sigma, \ldots$, then the enveloping algebra is generated by monomials $X_\rho, X_\rho X_\sigma, X_\rho X_\sigma X_\tau, \ldots$, which have various irreducible parts, obtained by appropriate symmetrisation. However, since any monomial $X_\rho \ldots X_\sigma \ldots$ can be re-ordered, by using the commutation relations, as $X_\rho X_\sigma \ldots$ plus lower order terms, it is clear that any anti-symmetrisation must reduce the degree (formally, the enveloping algebra is the free algebra of $X_\rho, X_\sigma, \ldots$ factored by the ideal generated by $X_\rho X_\sigma - X_\sigma X_\rho - [X_\rho, X_\sigma]$). Thus it is clear that the irreducible constituents of the enveloping algebra transform as symmetrised Kronecker powers of the adjoint representation. For the simple classical Lie algebras, the lowest symmetrised powers of the adjoint representation are given in table 1 (here and subsequently, we follow the notation of King 1975). Despite this apparent limitation, there are often physical arguments (for example, that the symmetry breaking term should couple symmetrically, in order for a Lagrangian to be constructed) which yield the same type of restriction.

Table 1. Lowest symmetrised powers of the adjoint for SU(n), O(n) and Sp(n).

Group and adjoint	Power	Irreducible constituents
SU(n)	2	$\{0\}+\{\bar{1};1\}+\{\bar{1}^2;1^2\}+\{\bar{2};2\}$
$\{\bar{1};1\}$	3	$\{0\}+2\{\bar{1};1\}+\{\bar{1}^2;2\}+\{\bar{2};1^2\}+\{\bar{2};2\}+\{\bar{1}^2;1^2\}$
		$+\{\overline{21};21\}+\{\bar{1}^3;1^3\}+\{\bar{3};3\}$
O(n)	2	$[0]+[2]+[2^2]+[1^4]$
$[1^2]$	3	$2[1^2]+[31]+[3^2]+[21^2]+[2^21^2]+[1^4]$
Sp(n)	2	$\langle 0\rangle+\langle 1^2\rangle+\langle 2^2\rangle+\langle 4\rangle$
$\langle 2\rangle$	3	$2\langle 2\rangle+\langle 21^2\rangle+\langle 2^3\rangle+\langle 31\rangle+\langle 42\rangle+\langle 4\rangle$

We end this section by giving a consistent notation for working with the generators and higher-order covariants of the classical Lie algebras (Bracken and Green 1971, Green 1971).

We start with a set of U(n) generators, $E_A{}^B$, $A, B = 1, \ldots, n$; with the commutation relations

$$[E_A{}^B, E_C{}^D] = \delta^B{}_C E_A{}^D - \delta_A{}^D E_C{}^B \tag{4}$$

in a representation such that $(E_A{}^B)^T = E_B{}^A$. The SU(n) generators $A_A{}^B$ are defined as

$$A_A{}^B = E_A{}^B - (1/n)\delta_A{}^B E_C{}^C \tag{5}$$

(with a summation convention for repeated indices), and again satisfy the commutation relations (4). Next, we introduce a metric tensor g_{AB}, $A, B = 1, \ldots, n$, and the generators

$$\Omega_{AB} = E_A{}^C g_{CB} - \eta E_B{}^C g_{CA} \tag{6}$$

where $g_{AB} = \eta g_{BA}$, $\eta = +1$ for O(n), and -1 for Sp(n). The Ω_{AB} satisfy the commutation relations

$$[\Omega_{AB}, \Omega_{CD}] = g_{CB}\Omega_{AD} - g_{AC}\Omega_{BD} + g_{DB}\Omega_{CA} - g_{AD}\Omega_{CB}. \tag{7}$$

These also serve for the transformation properties of the higher-order tensor operators. For example, by using an inverse metric tensor g^{AB}, where

$$g_{AC}g^{CB} = \delta_A{}^B,$$

(and n must be even, for Sp(n)), we can define the matrix powers

$$\Omega^{(1)}{}_{CD} = \Omega_{CD}, \qquad \Omega^{(2)}{}_{CD} = \Omega_{CE}g^{EF}\Omega_{FD}, \tag{8}$$

and so on. These transform under Ω_{AB} in the same way as Ω_{CD}, as in (7).

Finally, it is clear from the commutation relations that the traces of the matrix powers, $C_p = X^{(p)A}_A$ (where $X_A{}^B$ is any of $E_A{}^B$, $A_A{}^B$ or $\Omega_{AC}g^{CB}$), are invariants, and generate a complete set of algebraic invariants of G (the Casimir operators). The eigenvalues of the C_p are given in a convenient form by, for example, Rashid and Nwachuku 1978, and Edwards 1978.

In table 2 we collect together the above formulae for the generators and commutation relations of U(n), SU(n), O(n) and Sp(n), and give a convenient choice of metric for O(n) and Sp(n). Also given in each case are a set of diagonal generators (which will define the weights), and the eigenvalue of the second Casimir invariant C_2 (which is all we require in the following), in terms of the highest weight labels of the irreducible representation.

We are now ready to apply the ideas and tools developed in this section to the construction of tensor operators and their matrix elements. We turn firstly to the expectation values.

Table 2. Notation for classical Lie algebras.

	U(n)	SU(n)
X	$E_A{}^B$	$A_A{}^B = E_A{}^B - (1/n)\delta_A{}^B E_C{}^C$
H	$E_A{}^A, A=1,\ldots,n$	$A_A{}^A, A=1,\ldots,n$
[,]	$[E_A{}^B E_C{}^D] = \delta^B{}_C E_A{}^D - \delta_A{}^D E_C{}^B$	$[A_A{}^B A_C{}^D] = \delta^B{}_C A_A{}^D - \delta_A{}^D A_C{}^B$
$X^\dagger$	$(E_A{}^B)^\dagger = E_B{}^A$	$(A_A{}^B)^\dagger = A_B{}^A$
(λ)	$\{\lambda_1,\ldots,\lambda_n\}$	$\{\lambda_1,\ldots,\lambda_{n-1}\}$
C_2	$E_A{}^B E_B{}^A$	$A_A{}^B A_B{}^A$
C_2	$\sum_{i=1}^{n} \lambda_i(\lambda_i + n + 1 - 2i)$	$\sum_{i=1}^{n-1} \lambda_i(\lambda_i + b + 1 - 2i) - (1/n)\left(\sum_{i=1}^{n-1} \lambda_i\right)^2$
	O(n)	Sp(n)
X	$\Sigma_{AB} = (E\delta)_{AB} - (E\delta)_{BA}$	$\Xi_{AB} = (EJ)_{AB} + (EJ)_{BA}$
G	δ_{AB}	$J_{AB} = \delta_{A+(n/1)\,B} - \delta_{A\,B+(n/2)}$
H	$\mathrm{i}\Sigma_{2A-1\,2A}, A=1,\ldots,[n/2]$	$\Xi_{A\,A+n/2}, A=1,\ldots,n/2$
[,]	$[\Sigma_{AB}\Sigma_{CD}] = \delta_{AD}\Sigma_{BC} + \delta_{BC}\Sigma_{AD} - \delta_{BD}\Sigma_{AC} - \delta_{AC}\Sigma_{BD}$	$[\Xi_{AB}\Xi_{CD}] = -J_{AD}\Xi_{BC} - J_{BC}\Xi_{AD} - J_{AC}\Xi_{BD} - J_{BD}\Xi_{AC}$
$X^\dagger$	$\Sigma^\dagger_{AB} = \Sigma_{BA} = -\Sigma_{AB}$	$\Xi^\dagger_{AB} = \Xi^{AB} = \Xi^{BA}$
(λ)	$[\lambda_1,\ldots,\lambda_{[n/2]}]$	$\langle\lambda_1,\ldots,\lambda_{n/2}\rangle$
C_2	$\Sigma_{AB}\delta^{BC}\Sigma_{CD}\delta^{DA}$	$\Xi_{AB}J^{BC}\Xi_{CD}J^{DA}$
C_2	$2\sum_{i=1}^{[n/2]} \lambda_i(\lambda_i + n - 2i)$	$2\sum_{i=1}^{n/2} \lambda_i(\lambda_i + n + 2 - 2i)$

Table 3. Normalisation of generators of U(n), SU(n), O(n) and Sp(n).

	Generators	Normalisation
U(n)	$E_A{}^B$	$\langle\vert E_A{}^B E_C{}^D\vert\rangle = 2\delta_A{}^D\delta^B{}_C$
SU(n)	$A_A{}^B = E_A{}^B - (1/n)\delta_A{}^B E_C{}^C$	$\langle\vert A_A{}^B A_C{}^D\vert\rangle = 2(\delta_A{}^D\delta^B{}_C - (1/n)\delta_A{}^B\delta_C{}^D)$
O(n)	$\Sigma_{AB} = (E\delta)_{AB} - (E\delta)_{BA}$	$\langle\vert\Sigma_{AB}\Sigma_{CD}\vert\rangle = 4(\delta_{AD}\delta_{BC} - \delta_{AC}\delta_{BD})$
Sp(n)	$\Xi_{AB} = (EJ)_{AB} + (EJ)_{BA}$	$\langle\vert\Xi^{AB}\Xi_{CD}\vert\rangle = 4(\delta^A{}_D\delta^B{}_C + \delta^A{}_C\delta^B{}_D)$

4. Expectation values of tensor operators

Let us assume that, in a given broken symmetry situation, we have chosen, on physical or other grounds, that the symmetry breaking term V, should have a little group K, by hypothesis a subgroup of G. Suppose we wish to construct the expectation values of V in a certain irreducible representation, $\{\alpha\}$, of G. We proceed as follows:

Algorithm 1

1 The candidates for irreducible symmetry breaking terms are precisely those irreducible constituents, $\{\beta\}$ say, coupling to $\{\alpha\}\times\{\alpha\}$ which contain one or more K singlet parts, $\{0\}_0$

2 For each $\{\beta\}$, execute steps 2.1 to 2.3

 2.1 From the list of symmetrised powers of the adjoint representation, table 1, identify in what orders $\{\beta\}$ occurs. If there are more occurrences than the multiplicity of $\{\alpha\}$ in $\{\beta\}\times\{\alpha\}$, then the highest order ones are redundant.

 2.2 For each order, say (p), construct the irreducible tensor operator $V^{(p)}\sim\{\beta\}$. (If the generators are written in matrix form $X_A{}^B$, this step is straightforward.)

 2.3 For each orthogonal K-singlet $\{0\}_0$, execute steps 2.3.1 to 2.3.2.

  2.3.1 Project out the K-singlet component.

  2.3.2 Try to rearrange the expression for it into a linear combination of independent K-invariants (Casimir invariants and other non-subgroup invariants), with numerical coefficients.

 Return to 2.3 until completed.

 Return to 2 until completed.

3 The most general form of V is an arbitrary linear combination of expressions of the form 2.3.2.

This algorithm should be read in conjunction with the following general points. In step 1, there may be independent grounds (for example, the symmetry type of the couplings) for rejecting some candidates. In step 2.1, by hypothesis, at least one symmetrized power of the adjoint contains $\{\beta\}$. Otherwise, the method is obviously inapplicable.

As explained in connection with the order of the polynomial representing an adjoint operator, if $\{\alpha\}$ is non-generic, then some of the (higher order) occurrences of $\{\beta\}$ in symmetrised powers of the generator, will be redundant, in the irreducible representation $\{\alpha\}$. Finally, in step 2.3.2, the eigenvalue of the rearranged expression can, in principle, be written down immediately, in an arbitrary representation, in terms of the various highest weight, and non-subgroup, labels.

The total number of reduced matrix elements, in the most general form of V, will be $\Sigma_{\{\beta\}}\Sigma_{(p)}\Sigma_{\{0\}_0}1$. Thus, if $\{\alpha\}$ has some larger number of K-submultiplets, we obtain an appropriate number of constraints on the expectation values of V (Feldman and Matthews 1979). In practice, however, it is often unnecessary to consider the most general form. A general 'rule of thumb' seems to be that 'higher order' occurrences of a given symmetry breaking term in the enveloping algebra (in the sense of symmetrised powers of the adjoint), are 'higher order' in the sense of the smallness of their contributions.

We now illustrate this algorithm with some examples.

4.1. *Example* 1 $\mathrm{SU}(n)\supset\mathrm{SU}(n-1)\times U(1)$

Consider a system with symmetry SU(n) broken to SU($n-1$)$\times$U(1). We have

$$\{\bar{1}\}\times\{1\}=\{\bar{1};1\}+\{0\}$$

$$\{\bar{1};1\}\times\{\bar{1};1\}=(\{0\}+\{\bar{1};1\}+\{\bar{1}^2;1^2\}+\{\bar{2};2\})_{\mathrm{S}}+(\{\bar{1};1\}+\{\bar{1}^2;2\}+\{\bar{2};1^2\})_{\mathrm{A}}.$$

For this example, we shall be interested in symmetry breaking terms coupling symmetrically to $\{\bar{1}; 1\}$. We have the following reductions from SU(n) to SU($n-1$)$\times$U(1):

$$
\begin{aligned}
&\{1\} = \{1\}_{1/n} + \{0\}_{-(n-1)/n} \\
&\{\bar{1}; 1\} = \{\bar{1}\}_{-1} + \{0\}_0 + \{\bar{1}; 1\}_0 + \{1\}_1 \\
&\{\bar{1}^2; 1^2\} = \{\bar{1}^2; 1\}_{-1} + \{\bar{1}^2; 1^2\}_0 + \{\bar{1}; 1^2\}_1 \\
&\{\bar{2}; 2\} = \{\bar{2}\}_{-2}\{\bar{2}; 1\}_{-1} + \{\bar{1}\}_{-1} + \{\bar{2}; 2\}_0 + \{\bar{1}; 1\}_0 + \{0\}_0 + \{1\}_1\{\bar{1}; 2\}_1 + \{2\}_2.
\end{aligned}
\tag{9}
$$

Therefore, the available SU($n-1$)$\times$U(1) singlets (apart from the overall singlet) coupling symmetrically to $\{\bar{1}; 1\}$ are in $\{\bar{1}; 1\}$ and $\{\bar{2}; 2\}$.

From table 1, we see that $\{\bar{1}; 1\}$ occurs once in order one and two, and in general twice in order three of symmetrised powers of the adjoint (that is, $\{\bar{1}; 1\}$). For $n = 3$, the cubic terms are not independent, even in a generic irreducible representation. Thus, the order one and two occurrences are always sufficient: the result is the Gell-Mann Okubo formula (Gell-Mann 1962, Okubo 1962). The quadratic generalisation (see below) is still sufficient for the adjoint, but in a generic representation, higher order terms are required.

Also from table 1, we see that $\{\bar{2}; 2\}$ occurs once in order two, and once in order three, in symmetrised powers of the adjoint. However, since there is only one coupling to $\{\bar{1}; 1\}$ and the quadratic occurrence suffices. In general, however, higher-order terms may occur.

Let us carry out, for the $\{\bar{2}; 2\}$ operator of symmetrised order 2, the remaining steps of the algorithm. Firstly (step 2.2) we construct the tensor operator $\{\bar{2}; 2\}$. In terms of the generators $A_A{}^B$ of SU(n) (table 2), it is

$$
W_{AB}{}^{CD} = X_{AB}{}^{CD} - (n+2)^{-1} Y_{AB}{}^{CD} + [(n+1)(n+2)]^{-1} Z_{AB}{}^{CD}
$$

where

$$
\begin{aligned}
X_{AB}{}^{CD} &= \{A_A{}^C, A_B{}^D\} + \{A_A{}^D, A_B{}^C\} \\
Y_{AB}{}^{CD} &= \delta_A{}^C X_{EB}{}^{ED} + \delta_B{}^C X_{EA}{}^{ED} + \delta_A{}^D X_{EB}{}^{EC} + \delta_B{}^D X_{EA}{}^{EC} \\
Z_{AB}{}^{CD} &= (\delta_A{}^C \delta_B{}^D + \delta_A{}^D \delta_B{}^C)\{A_E{}^F, A_F{}^E\}
\end{aligned}
\tag{10}
$$

and {, } stands for the anticommutator. Next, (step 2.3.1), we must project out the SU($n-1$)$\times$U(1) singlet piece. For completeness, and for later use, the reductions of both $\{\bar{1}; 1\}$, $A_A{}^B$, and $\{\bar{2}; 2\}$, $W_{AB}{}^{CD}$, into their common irreducible pieces under SU($n-1$)$\times$U(1), are given in table 4. Some additional notation is also introduced there.

It remains (step 2.3.2) to rearrange the expression for the SU($n-1$)$\times$U(1) singlet $Z_n{}^W$ (table 4) in terms of the generating system of invariants. The result, after straightforward algebraic manipulation, is

$$
V \sim \left| \begin{matrix} \{\bar{2}; 2\} \\ \{0\}_0 \end{matrix} \right\rangle = a_1(2n^2(n+1)Z_n{}^2 - n(n-1)C_2\{n\} + (n+1)(n-1)C_2\{n-1\}).
\tag{11}
$$

note that the dimension of $\{\bar{2}; 2\}$ is

$$
d^{\{\bar{2};2\}} = \tfrac{1}{4} n^2 (n-1)(n+3).
$$

Table 4. Common submultiplets of $\mathrm{Su}(n-1)\times \mathrm{U}(1)$ in $\{\bar{1};1\}$ and $\{\bar{2};2\}$ of $\mathrm{SU}(n)$.

Labelling chain	$\mathrm{SU}(n)\supset \mathrm{SU}(n-1)\times \mathrm{U}(1)_{Z_n}\supset \mathrm{SU}(n-2)\times \mathrm{U}(1)_{Z_{n-1}}\times \mathrm{U}(1)_{Z_n}\supset\ldots$					
Notation for states	$\left\vert \begin{matrix} \mathrm{SU}(n-1)_{Z_n} \\ \mathrm{SU}(n-2)_{Z_{n-1}} \\ \ldots \end{matrix} \right\rangle$					
Submultiplet state	$\{\bar{1};1\}$	N^2	ϕ	$\{\bar{2};2\}$	N^2	ϕ
$\left\vert \begin{matrix} \{1\}_1 \\ \{0\}_{-1+(n-1)^{-1}} \end{matrix} \right\rangle$	${A_{n-1}}^{n}$	2	+	${W_{n-1\,n}}^{n\,n}$	$32\dfrac{n}{n+2}$	+
$\left\vert \begin{matrix} \{\bar{1};1\}_0 \\ \{1\}_1 \\ \{0\}_{-(n-2)^{-1}} \end{matrix} \right\rangle$	${A_{n-2}}^{n-1}$	2	−	${W_{n-2\,n}}^{n-1\,n}$	$16\dfrac{n+1}{n+2}$	−
$\left\vert \{0\}_0 \right\rangle$	$Z_n=-{A_n}^{n}$	$2\dfrac{n-1}{n}$	−	${Z_n}^{w}=-{W_{n\,n}}^{n\,n}$	$64\dfrac{n((n-1)}{(n+1)(n+2)}$	−
$\left\vert \begin{matrix} \{\bar{1}\}_{-1} \\ \{0\}_{1-(n-1)^{-1}} \end{matrix} \right\rangle$	${A_n}^{n-1}$	2	−	${W_{n\,n}}^{n-1\,n}$	$32\dfrac{n}{n+2}$	−

The form of result (11) can be compared with that of the $\mathrm{SU}(n)$ (quadratic) Gell-Mann Okubo formula, which can be obtained in the same way. It is

$$V\sim\left\vert \begin{matrix} \{\bar{1};1\} \\ \{0\}_0 \end{matrix} \right\rangle = a_2 Z_n + a_3(n(n-2){Z_n}^2+(n-1)(n-2)C_2\{n\}-n(n-1)C_2\{n-1\}) \qquad (12)$$

and $d^{\{\bar{1};1\}}=n^2-1$. Here a_1, a_2 and a_3 are arbitrary constants.

4.2. *Example* 2 $\mathrm{O}(n)\supset \mathrm{O}(n-2)\times \mathrm{U}(1)$

Consider a system with a symmetry $\mathrm{O}(n)$ broken to $\mathrm{O}(n-2)\times\mathrm{U}(1)$. We wish to construct symmetry breaking terms coupling symmetrically to the fundamental representation [1]. We have

$$[1]\times[1]=([0]+[2])_S+[1^2]_A \qquad (13)$$

and the following reductions from $\mathrm{O}(n)$ to $O(n-2)\times \mathrm{U}(1)$:

$$\begin{aligned} &[1]=[0]_{-1}+[1]_0+[0]_1 \\ &[1^2]=[1]_{-1}+[1^2]_0+[0]_0+[1]_1 \qquad (14) \\ &[2]=[0]_{-2}+[1]_{-1}+[2]_0+[0]_0+[1]_1+[0]_2. \end{aligned}$$

Hence the only available symmetry breaking term coupling symmetrically is [2]. From table 1 this occurs once in symmetrised order two, not in symmetrised order three, and again in symmetrised order four, in the enveloping algebra. However, for the fundamental,

$$[2]\times[1]=[1]+[21]+[3],$$

so that there is only one coupling, and the quadratic suffices. Again, in general higher orders may be required; however, for $n \leqslant 8$, the fourth order terms are redundant, and the quadratic suffices even for a generic representation.

Let us carry out for the [2] of symmetrised order two, the remaining steps of the algorithm. Firstly (step 2.2), we construct the tensor operator [2]. In terms of the O(n) generators Σ_{AB} (table 2), it is

$$M_{AB}=\{\Sigma_{AC}\Sigma_{CB}\}-(1/n)\delta_{AB}\{\Sigma_{CD}\Sigma_{DC}\} \tag{15}$$

Next (step 2.3.1), we must project out the $\mathrm{O}(n-2)\times\mathrm{U}(1)$ singlet piece. Once again, for completeness and for later use, the reductions of both $[1^2]$, Σ_{AB} and [2], M_{AB} into their common irreducible parts, under $\mathrm{O}(n-2)\times\mathrm{U}(1)$, are given in table 5, together with some additional notation.

It remains (step 2.3.2) to rearrange the expression for the $\mathrm{O}(n-2)\times\mathrm{U}(1)$ singlet component (table 5) $H^M_{[n/2]}$ in terms of the independent invariants. The result, after straightforward algebraic manipulation, is

$$V\sim\left|\begin{matrix}[2]\\ [0]_0\end{matrix}\right\rangle=a((n-4)C_2[n]-nC_2[n-2]+2nH^2_{[n/2]}) \tag{16}$$

where a is an unknown constant. Note that

$$d^{[2]}=\tfrac{1}{2}(n-1)(n+2).$$

4.3. *Example* 3 $\mathrm{Sp}(2p+2q)\supset\mathrm{Sp}(2p)\times\mathrm{Sp}(2q)$

Consider a system with symmetry $\mathrm{Sp}(2p+2q)$ broken to $\mathrm{Sp}(2p)\times\mathrm{Sp}(2q)$. We wish to

Table 5. Common submultiplets of $\mathrm{O}(n-2)\times\mathrm{U}(1)$ in $[1^2]$ and [2] of O(n).

Labelling chain	$\mathrm{O}(n)\supset\mathrm{O}(n-2)\times\mathrm{U}(1)\supset\mathrm{O}(n-4)\times\mathrm{U}(1)\times\mathrm{U}(1)\supset\ldots$					
Notation for states	$\left\lvert\begin{matrix}\mathrm{O}(n-2)_{H[n/2]}\\ \mathrm{O}(n-4)_{H[n/2]-1}\\ \ldots\end{matrix}\right\rangle$					
Definitions	$\Sigma^{+-}=\Sigma_{n-3\,n-1}+\mathrm{i}\Sigma_{n-3\,n}-\mathrm{i}\Sigma_{n-2\,n-1}+\Sigma_{n-2\,n}$ $\Sigma^{-+}=\Sigma_{n-3\,n-1}-\mathrm{i}\Sigma_{n-3\,n}+\mathrm{i}\Sigma_{n-2\,n-1}+\Sigma_{n-2\,n}$ $H_{[n/2]}=\mathrm{i}\Sigma_{n-1\,n}$ $H^M_{[n/2]}=M_{n-1\,n-1}+M_{nn}$					
Submultiplet state	$[1^2]$	N^2	ϕ	[2]	N^2	ϕ
$\left\lvert\begin{matrix}[1]_1\\ [0]_{-1}\end{matrix}\right\rangle$	Σ^{-+}	16	+	M^{-+}	$128(n-2)$	+
$\left\lvert\begin{matrix}[0]_0\\ [0]_0\end{matrix}\right\rangle$	$H_{[n/2]}=\mathrm{i}\Sigma_{n-1\,n}$	4	+	$H^M_{[n/2]}$	$128\dfrac{(n-2)^2}{n}$	−
$\left\lvert\begin{matrix}[1]_{-1}\\ [0]_n\end{matrix}\right\rangle$	Σ^{+-}	16	+	M^{+-}	$128(n-2)$	−

construct symmetry breaking terms coupling to the fundamental representation $\langle 1\rangle$. We have

$$\langle 1\rangle\times\langle 1\rangle=\langle 2\rangle_s+(\langle 1^2\rangle+\langle 0\rangle)_{\mathrm{A}}, \tag{17}$$

and the following reductions from $\mathrm{Sp}(2p+2q)$ to $\mathrm{Sp}(2p)\times\mathrm{Sp}(2q)$:

$$\begin{aligned}
\langle 1\rangle&=\langle 1\rangle\times\langle 0\rangle+\langle 0\rangle\times\langle 1\rangle\\
\langle 2\rangle&=\langle 2\rangle\times\langle 0\rangle+\langle 1\rangle\times\langle 1\rangle+\langle 0\rangle\times\langle 2\rangle\\
\langle 1^2\rangle&=\langle 1^2\rangle\times\langle 0\rangle+\langle 1\rangle\times\langle 1\rangle+\langle 0\rangle\times\langle 0\rangle+\langle 0\rangle\times\langle 1^2\rangle.
\end{aligned} \tag{18}$$

Hence the only available symmetry breaking term is $\langle 1^2\rangle$ (which actually couples antisymmetrically to $\langle 1\rangle$). From table 1, this occurs once in symmetrised order two, not in symmetrised order three, (and again in symmetrised order four, and so on), in the enveloping algebra. However, for the coupling to $\langle 1\rangle$,

$$\langle 1^2\rangle\times\langle 1\rangle=\langle 21\rangle+\langle 1^3\rangle+\langle 1\rangle$$

so that there is only one coupling, and the quadratic suffices. As before, in general, higher orders may be required; however, for $n\leqslant 8$, the fourth order terms are redundant, and the quadratic suffices, even for a generic representation.

Let us carry out for the $\langle 1^2\rangle$ of symmetrised order 2, the remaining steps of the algorithm. Firstly (step 2.2), we construct the tensor operator $\langle 1^2\rangle$. In terms of the $\mathrm{Sp}(2n)$ generators Ξ_{AB} (table 2), it is

$$Q_{AB}=J^{CD}\{\Xi_{AC},\Xi_{DB}\}-n^{-1}J_{AB}\Xi^2{}_c{}^c. \tag{19}$$

Next (step 2.3.1), we project out the $\mathrm{Sp}(2p)\times\mathrm{Sp}(2q)$ singlet piece. Again, for completeness, and for later comparison, the complete reductions of both $\langle 2\rangle$, Ξ_{AB}, and $\langle 1^2\rangle$, Q_{AB}, into their common irreducible constituents under $\mathrm{Sp}(2p)\times\mathrm{Sp}(2q)$, are given in table 6, together with some additional notation.

It remains (step 2.3.2) to rearrange the expression for the $\mathrm{Sp}(2p)\times\mathrm{Sp}(2q)$ singlet component $\check{Z}^Q$ (table 6) in terms of the independent invariants. The result, after straightforward algebraic manipulation, is

$$V\sim\left|\begin{matrix}\langle 1^2\rangle\\ \langle 0\rangle\times\langle 0\rangle\end{matrix}\right\rangle=a[(p-q)C_2\langle 2n\rangle+n(C_2\langle 2q\rangle-C_2\langle 2p\rangle)], \tag{20}$$

where $n=p+q$, and a is arbitrary. Finally, note $d^{\langle 1^2\rangle}=(n-1)(2n+1)$.

4.4. *Example* 4 $\mathrm{Sp}(2n)\supset\mathrm{SU}(n)\times\mathrm{U}(1)$

Consider a system with symmetry $\mathrm{Sp}(2n)$ broken to $\mathrm{SU}(n)\times\mathrm{U}(1)$. We wish to construct symmetry breaking terms coupling to the fundamental representation $\langle 1\rangle$. Again we have (17), and the $\mathrm{Sp}(2n)\supset\mathrm{SU}(n)\times\mathrm{U}(1)$ reductions are

$$\begin{aligned}
\langle 1\rangle&=\{1\}_1+\{\bar{1}\}_{-1}\\
\langle 1^2\rangle&=\{\overline{1^2}\}_{-2}+\{\bar{1};1\}_0+\{1^2\}_2\\
\langle 2\rangle&=\{\bar{2}\}_{-2}+\{\bar{1};1\}_0+\{0\}_0+\{2\}_2.
\end{aligned} \tag{21}$$

Table 6. Common submultiplets of $\mathrm{SU}(p)\times\mathrm{U}(1)\times\mathrm{SU}(q)\times\mathrm{U}(1)$ in $\langle 2\rangle$ and $\langle 1^2\rangle$ of $\mathrm{Sp}(2p+2q)$.

Labelling chain	$\mathrm{Sp}(2p+2q)\supset\mathrm{Sp}(2p)\times\mathrm{Sp}(2q)\supset\mathrm{SU}(p)\times\mathrm{U}(1)\times\mathrm{SU}(q)\times\mathrm{U}(1)\supset\ldots$					
Notation for states	$\left\lvert\begin{matrix}\mathrm{Sp}(2p)\times\mathrm{Sp}(2q)\\ \mathrm{SU}(p)_{\tilde{Z}}\times\mathrm{SU}(q)_{\tilde{X}}\\ \mathrm{SU}(p-1)_{Z_p}\times\mathrm{SU}(q-1)_{X_q}\\ \ldots\end{matrix}\right\rangle^{\mathrm{Sp}(2n)}$					
Definitions	$\tilde{Z}=\sum_{K=1}^{p}\Xi_{K\,K+n}$ $\tilde{X}=\sum_{\alpha=p+1}^{n}\Xi_{\alpha\,\alpha+n}$ $Z_p=-(\Xi_{p\,p+n}-p^{-1}\tilde{Z})$ $X_q=-(\Xi_{q\,q+n}-q^{-1}\tilde{X})$			$\tilde{Z}^Q=\sum_{K=1}^{p}Q_{\kappa\,\kappa+n}$ $Z_p^Q=-(Q_{p\,p+n}-p^{-1}\tilde{Z}^Q)$ $X_q^Q=-(Q_{n\,2n}+q^{-1}\tilde{Z}^Q)$		
Submultiplet state	$\langle 2\rangle$	N^2	ϕ	$\langle 1^2\rangle$	N^2	ϕ
$\left\lvert\begin{matrix}\{1\}_1\times\{1\}_1\\ \{0\}_{-1+p^{-1}}\times\{0\}_{-1+q^{-1}}\end{matrix}\right\rangle$	$\Xi^{++}=\Xi_{p\,n}$	4	+	$Q^{++}=Q_{p\,n}$	$64(n+1)$	+
$\left\lvert\begin{matrix}\{1\}_1\times\{\bar{1}\}_{-1}\\ \{0\}_{-1+p^{-1}}\times\{0\}_{1-q^{-1}}\end{matrix}\right\rangle$	$\Xi^{+-}=\Xi_{p\,2n}$	4	+	$Q^{+-}=Q_{p\,2n}$	$64(n+1)$	+
$\left\lvert\begin{matrix}\{\bar{1}1\}_0\times\{0\}_0\\ \{0\}_0\times\{0\}_0\end{matrix}\right\rangle$	Z_p	$4\left(\frac{p-1}{p}\right)$	+	Z_p^Q	$64(n+1)\left(\frac{p-1}{p}\right)$	+
$\left\lvert\begin{matrix}\{0\}_0\times\{0\}_0\\ \{0\}_0\times\{0\}_0\end{matrix}\right\rangle$	$\tilde{Z}$ $\tilde{X}$	$4p$ $4q$	− −	$\tilde{Z}^Q$	$64\left(\frac{n+1}{n}\right)pq$	−
$\left\lvert\begin{matrix}\{0\}_0\times\{\bar{1}1\}_0\\ \{0\}_0\times\{0\}_0\end{matrix}\right\rangle$	X_q	$4\left(\frac{q-1}{q}\right)$	+	X_q^Q	$64(n+1)\left(\frac{q-1}{q}\right)$	−
$\left\lvert\begin{matrix}\{\bar{1}\}_{-1}\times\{\bar{1}\}_{-1}\\ \{0\}_{1-p^{-1}}\times\{0\}_{1-q^{-1}}\end{matrix}\right\rangle$	$\Xi^{--}=\Xi_{p+n\,2n}$	4	+	$Q^{--}=Q_{p+n\,2n}$	$64(n+1)$	−
$\left\lvert\begin{matrix}\{\bar{1}\}_{-1}\times\{1\}_1\\ \{0\}_{1-p^{-1}}\times\{0\}_{-1+q^{-1}}\end{matrix}\right\rangle$	$\Xi^{-+}=\Xi_{p+n\,n}$	4	−	$Q^{-+}=Q_{p+n\,n}$	$64(n+1)$	+

Hence the only available symmetry breaking term is $\langle 2\rangle$ (which couples symmetrically to $\langle 1\rangle$). From table 1, this occurs once in symmetrised order one, not in symmetrised order two, and in general twice in symmetrised order three, and so on, in the enveloping algebra. For the coupling to $\langle 1\rangle$,

$$\langle 2\rangle\times\langle 1\rangle=\langle 3\rangle+\langle 21\rangle+\langle 1\rangle,$$

there is only one coupling, and the linear term suffices. As before, in general, higher orders may be required.

Let us carry out, for the $\langle 2\rangle$ of (symmetrised) order one, the remaining steps of the algorithm. Obviously (step 2.2), the tensor operator is Ξ_{AB}. For the reductions to $\mathrm{SU}(n)\times\mathrm{U}(1)$ (step 2.3.1), table 6 gives the common irreducible constituents of both $\langle 2\rangle$, Ξ_{AB}, and $\langle 1^2\rangle$, Q_{AB} respectively.

Step 2.3.2 is trivial, since the singlet component $\tilde{Z}$ (table 6) is already one of the independent invariants of $SU(n)\times U(1)$. Thus we have

$$V \sim \left| \begin{matrix} \langle 2 \rangle \\ \{0\}_0 \end{matrix} \right\rangle = a\tilde{Z}, \tag{22}$$

where a is arbitrary. Finally, note that

$$d^{\langle 2 \rangle} = n(2n+1).$$

5. Transition matrix elements of tensor operators:

In this section, we extend our formalism to encompass the calculation of transition matrix elements of the symmetry breaking terms (tensor operators). The major new ingredient required is the introduction of a norm for the enveloping algebra.

We take the $U(n)$ generators $E_A{}^B$ (table 2) in a (reducible) representation with a cyclic state $|\,\rangle$, which is a $U(n)$ singlet. The n^2 states $E_A{}^B|\,\rangle$ clearly transform like the adjoint representation, $\{\bar{1}\}\times\{1\}$. We shall assume that they are orthogonal states, of length 2:

$$\langle|E_A{}^B E_C{}^D\rangle\rangle = \langle|(E_B{}^A)^{+} E_C{}^D|\rangle = 2\delta_A{}^D\delta^B{}_C. \tag{23}$$

Similarly, the states $E_A{}^B E_C{}^D|\rangle$ transform like $(\{\bar{1}\}\times\{1\})^2$, and have the normalisation

$$\langle|E_A{}^B E_C{}^D E_E{}^F E_G{}^G|\rangle = 4\delta_A{}^H\delta^B{}_G\delta_C{}^F\delta^D{}_E. \tag{24}$$

This can obviously be extended to any monomial in the generators.

This also specifies the normalisation of the generators $A_A{}^B$, Σ_{AB} and Ξ_{AB}, of $SU(n)$, $O(n)$, and $Sp(n)$ respectively, which are defined in terms of the $E_A{}^B$ (table 2). These are given in table 3.

Using this realisation, we can now construct normalised states (up to a relative phase) of the admissible irreducible representations, explicitly in terms of the generators. As before, the admissible irreducible representations available to us are those in symmetrised powers of the adjoint (table 1). Before giving an algorithm for computing transition matrix elements, we must complete the specification of the normalised states by giving a phase convention.

Firstly, we choose in G a complete set of rank G independent diagonal generators H (for example, see table 2). We shall assume that, in an irreducible representation $\{\alpha\}$, a total ordering of states has been specified, which contains the usual lexical partial ordering by weights (since the irreducible representations are finite-dimensional, this can always be done). Also, we select a set of rank G independent weight lowering generators X_-.

Consider an admissible irreducible representation $\{\alpha\}$ of G. We can identify (up to a phase) the unique highest-weight state with a component of the normalised irreducible tensor:

$$\left| \begin{matrix} \{\alpha\} \\ HW \end{matrix} \right\rangle \propto V_{HW}|\rangle$$

The matrix elements of the X_- (and their hermitean conjugates X_+) between states of $\{\alpha\}$ may be obtained by commutation with the appropriate component of V. Our phase conventions for the states in $\{\alpha\}$ are in fact established by specifying the phases of the matrix elements of X_-, as follows:

Phase Conventions

1 The highest-weight state of $\{\alpha\}$ is assigned a positive phase:

$$\left|\begin{matrix}\{\alpha\}\\ HW\end{matrix}\right\rangle = +V_{HW}|\rangle. \tag{25}$$

2 For U(n) and SU(n), the matrix elements of each X_- (and X_+) are positive.

3 For O(n) and Sp(n), and for each X_-, the highest state (relative to the total ordering) which couples through X_- to a given state, has a positive matrix element.

Suppose we wish to calculate Clebsch–Gordan coefficients of the form

$$\left\langle\begin{matrix}\{\alpha\}\\ \{\kappa\}_X\end{matrix}\right|\begin{matrix}\{\beta\}\\ \{0\}_0\end{matrix};\begin{matrix}\{\gamma\}\\ \{\kappa\}_X\end{matrix}\right\rangle, \tag{26}$$

where $\{\alpha\}\neq\{\gamma\}$. That is, we wish to evaluate those singlet factors of K in $\{\alpha\}$ contained in $\{\beta\}\times\{\gamma\}$ of G, for which the K-submultiplet is a singlet (see (2)). Our procedure is the following:

Algorithm 2

1 The number of independent couplings is the multiplicity of the singlet $\{0\}$ in $\{\bar{\alpha}\}\times\{\beta\}\times\{\gamma\}$.

2 Establish the symmetrised orders in the enveloping algebra (table 2) in which $\{\alpha\}$, $\{\beta\}$ and $\{\gamma\}$ occur. The number of independent couplings should correspond to the independent combinations in which order $\{\alpha\}$ = order $\{\beta\}$ + order $\{\gamma\}$.

3 For each such coupling, execute steps 3.1 to 3.4.

   3.1 Construct the tensor operators corresponding to $\{\alpha\}$, $\{\beta\}$ and $\{\gamma\}$, in the appropriate order.

   3.2 For $\{\alpha\}$ and $\{\gamma\}$, and for each common K submultiplet $\{\kappa\}_X$, project out a normalised state, with the correct phase.

   3.3 For $\{\beta\}$ project out the normalised K-singlet state(s), with the correct phase(s).

   3.4 For each such singlet state, and for each state 3.2, form the inner product (26), and evaluate it using the table 3.

   Return to step 3 until completed.

In step 3.4, it is often easier to extract the required singlet factor by identifying the appropriate term in the expression for the particular component of $\{\alpha\}$, provided this term is orthogonal to the remainder of the expression.

We end this section with some examples to illustrate the algorithm.

5.1. *Example* 5 SU(n) ⊃ SU($n-1$) × U(1): $\{\bar{2}; 2\}$ *in* $\{\bar{1}; 1\}\times\{\bar{1}; 1\}$

From example 1, § 4.1, we know that there is only one coupling, corresponding to the occurrence of $\{\bar{2}; 2\}$ in symmetrised order two in the enveloping algebra (steps 1, 2).

We label states according to the subgroup chain (table 4).

$$\mathrm{SU}(n)\supset\mathrm{SU}(n-1)\times\mathrm{U}(1)_{Z_n}\supset\mathrm{SU}(n-2)\times\mathrm{U}(1)_{Z_{n-1}}\times\mathrm{U}(1)_{Z_n}\supset\ldots \tag{27}$$

where the orthogonal diagonal generators are

$$\begin{aligned} Z_n &= -A_n{}^n \\ Z_{n-1} &= -(A_{n-1}{}^{n-1} + (n-1)^{-1}A_n{}^n) \\ Z_{n-2} &= -[A_{n-2}{}^{n-2} + (n-2)^{-1}(A_{n-1}{}^{n-1} + A_{n-2}{}^{n-2})]. \end{aligned} \tag{28}$$

and so on. We take as the independent shifting operators the set

$$A_A{}^{A+1}, A = 1, \ldots, n-1.$$

In table 4 the tensors $W_{AB}{}^{CD}$ (see (10)) and $A_A{}^B$ are decomposed with respect to $\mathrm{SU}(n-1)\times \mathrm{U}(1)$ (only the common submultiplets are shown: cf (9)). States belonging to each submultiplet are shown, together with their normalisations N^2, and relative phases (up to an overall sign). The latter may be verified directly from the phase conventions, and the commutation relations (steps 3.2 and 3.3).

As an example of these steps, we have from (10) the following expression for $W^n_{n-1}{}^n_n$ (see table 4).

$$W^n_{n-1}{}^n_n = -\frac{2n^2}{(n-1)(n+2)}\{A^n_{n-1}Z_n\} - \frac{2}{n+2}\left\{A_{n-1}{}^a - \frac{1}{n-1}\delta_{n-1}{}^a Z_n, A_a{}^n\right\}$$

The first term contributes to the normalisation N^2 an amount

$$2\left(\frac{2n^2}{(n-1)(n+2)}\right)^2 . 2 . \left[2\left(\frac{n-1}{n}\right)\right],$$

the overall factor 2 coming from the anticommutator. From the second term with $a \neq n-1$, we have

$$2(n-2)\left(\frac{2}{(n+2)}\right)^2 . 2 . 2,$$

and with $a = n-1$, we have

$$2\left[\frac{2}{(n+2)}\right]^2 . \left[2\frac{n-2}{n-1}\right] . 2,$$

and since the terms are orthogonal, the result is

$$N^2 = \frac{32n}{(n-1)(n+2)}.$$

The singlet factor (and its correct relative phase) can now be written down from the first term, taking into account the normalisation of $A_{n-1}{}^n$ and Z_n (step 3.4).

The singlet factors thus evaluated are summarised in table 5(i). They can be checked, for the cases $n = 3$ (**27** contained in $\mathbf{8}\times\mathbf{8}$) and $n = 4$ (**84** contained in $\mathbf{15}\times\mathbf{15}$), by consulting published tables: see for example Haacke *et al* 1976.

5.2. *Example* 6 $\mathrm{O}(n) \supset \mathrm{O}(n-2)\times \mathrm{U}(1)$: [2] *in* $[1^2]\times[1^2]$:

There is in general only one coupling of $[1^2]\times[1^2]$ to [2], since

$$[1^2]\times[1^2] = [0]+[2]+[1^2]+[2^2]+[21^2]+[1^4].$$

Obviously, we take the $[1^2]$ factors linear (symmetrised order 1) in the generators.

Also, from table 1, it can be seen that [2] occurs once in symmetrised order 2 in the enveloping algebra, not in symmetrised order 3, and again in symmetrised order 4, and so on. Thus the quadratic is sufficient (steps 1, 2).

We label states according to the subgroup chain

$$\mathrm{O}(n) \supset \mathrm{O}(n-2) \times \mathrm{U}(1)_{H_{[n/2]}} \supset \mathrm{O}(n-4) \times \mathrm{U}(1)_{H_{[n/2]-1}} \times \mathrm{U}(1)_{H_{[n/2]}} \supset \ldots,$$

where the orthogonal diagonal generators are

$$H_{[n/2]} = \mathrm{i}\Sigma_{n-1\,n}, \qquad H_{[n/2]-1} = \mathrm{i}\Sigma_{n-3\,n-2}. \tag{29}$$

and so on. We take as shifting operator (see table 4(ii))

$$\Sigma^{+-} = \Sigma_{n-3\,n-1} + \mathrm{i}\Sigma_{n-3\,n} - \mathrm{i}\Sigma_{n-2\,n-1} + \Sigma_{n-2\,n} \tag{30}$$

in order to define the phases.

In table 5, the tensors M_{AB} (see (15)) and Σ_{AB} are decomposed with respect to $\mathrm{O}(n-2) \times \mathrm{U}(1)$ (only the common submultiplets are shown: cf (14)). States belonging to each submultiplet are shown, together with their normalisations N^2, and relative phases (up to an overall sign). The calculation of the normalisations N^2 proceeds as in the previous example (steps 3.2 and 3.3).

As an example of these steps, let us follow the determination of the relative phases in $\mathrm{O}(n)$. Using the shifting operator Σ^{+-} and (15), and in the notation of table 5, we have the following commutation relations:

$$[\Sigma^{+-}, M^{-+}] = 2H^M_{[n/2]-1} - [2n/(n-2)]H^M_{[n/2]},$$

$$[\Sigma^{+-}, H^M_{[n/2]}] = 2M^{+-},$$

$$[\Sigma^{+-}, H^M_{[n/2]-1}] = -2((n-4)/(n-2))M^{+-},$$

so that, using the phase conventions, with M^{-+} conventionally assigned a positive phase, the relative phases of $H^M_{[n/2]}$, $H^M_{[n/2]-1}$ and M^{+-} are $-$, $+$ and $-$, respectively. The same calculation, but with $\Sigma([1^2])$ instead of $M([2])$, gives, for Σ^{-+} conventionally positive, the relative phases of $H_{[n/2]}$, $H_{[n/2]-1}$ and Σ^{+-} to be $+$, $-$ and $+$.

The singlet factors thus evaluated are summarised in table 7(b). Again, they can be checked, in specific cases, by consulting published tables: for example, for $n = 8$, see Barnes *et al* 1978.

5.3. Example 7 $\mathrm{Sp}(2p+2q) \supset \mathrm{SU}(p) \times \mathrm{U}(1) \times \mathrm{SU}(q) \times \mathrm{SU}(1)$: $\langle 1^2 \rangle$ *in* $\langle 2 \rangle \times \langle 2 \rangle$

There is, in general, only one coupling of $\langle 2 \rangle \times \langle 2 \rangle$ to $\langle 1^2 \rangle$, since

$$\langle 2 \rangle \times \langle 2 \rangle = \langle 0 \rangle + \langle 1^2 \rangle + \langle 2 \rangle + \langle 2^2 \rangle + \langle 3 \rangle + \langle 4 \rangle.$$

Also, from table 1, it can be seen that $\langle 1^2 \rangle$ occurs once in symmetrised order two in the enveloping algebra, not in symmetrised order three, and again in symmetrised order four, and so on. Obviously, we take the $\langle 2 \rangle$ factors linear (symmetrised order 1) in the generators: thus, the quadratic occurrence is sufficient (steps 1, 2).

We label states according to the sub-group chain

$$\mathrm{Sp}(2p+2q) \supset \mathrm{Sp}(2p) \times \mathrm{Sp}(2q) \supset \mathrm{SU}(p) \times \mathrm{U}(1)_{\tilde{z}} \times \mathrm{U}(1)_{\tilde{x}} \supset \ldots,$$

and as in (27) within $\mathrm{SU}(p)$ and $\mathrm{SU}(q)$. The diagonal generators $\tilde{Z}$ and $\tilde{X}$ are defined by

$$\tilde{Z} = \sum_{K=1}^{p} \Xi_{K\,K+n}, \qquad \tilde{X} = \sum_{\alpha=1}^{q} \Xi_{p+\alpha\,p+n+\alpha}, \tag{31}$$

Table 7. (a). Singlet factors of SU($n-1$)$\times$U(1) for $\{\bar{1};1\}\times\{\bar{1};1\}$ contains $\{\bar{2};2\}$ of SU(n). (b). Singlet factors of O($n-2$)$\times$U(1) for $[1^2]\times[1^2]$ contains [2] of O(n).

(a)	$\left\langle \begin{matrix}\{\bar{2};2\}\\ \{1\}_1\end{matrix} \middle\vert \begin{matrix}\{\bar{1};1\}\\ \{0\}_0\end{matrix}; \begin{matrix}\{\bar{1};1\}\\ \{1\}_1\end{matrix}\right\rangle$	$+(1/2^{1/2})\left(\frac{n^2}{(n-1)(n+2)}\right)^{1/2}$
	$\left\langle \begin{matrix}\{\bar{2};2\}\\ \{\bar{1};1\}_0\end{matrix} \middle\vert \begin{matrix}\{\bar{1};1\}\\ \{0\}_0\end{matrix}; \begin{matrix}\{\bar{1};1\}\\ \{\bar{1};1\}_0\end{matrix}\right\rangle$	$+1/2\left(\frac{n(n+1)}{(n-1)(n+2)}\right)^{1/2}$
	$\left\langle \begin{matrix}\{\bar{2};2\}\\ \{0\}_0\end{matrix} \middle\vert \begin{matrix}\{\bar{1};1\}\\ \{0\}_0\end{matrix}; \begin{matrix}\{\bar{1};1\}\\ \{0\}_0\end{matrix}\right\rangle$	$+\left(\frac{n^3}{(n-1)(n+1)(n+2)}\right)^{1/2}$
	$\left\langle \begin{matrix}\{\bar{2};2\}\\ \{\bar{1}\}_{-1}\end{matrix} \middle\vert \begin{matrix}\{\bar{1};1\}\\ \{0\}_0\end{matrix}; \begin{matrix}\{\bar{1};1\}\\ \{\bar{1}\}_{-1}\end{matrix}\right\rangle$	$+(1/2^{1/2})\left(\frac{n^2}{(n-1)(n+2)}\right)^{1/2}$
(b)	$\left\langle \begin{matrix}[2]\\ [1]_1\end{matrix} \middle\vert \begin{matrix}[1^2]\\ [0]_0\end{matrix}; \begin{matrix}[1^2]\\ [1]_1\end{matrix}\right\rangle$	$+1/2^{1/2}\left[\frac{1}{n-2}\right]^{1/2}$
	$\left\langle \begin{matrix}[2]\\ [0]_0\end{matrix} \middle\vert \begin{matrix}[1^2]\\ [0]_0\end{matrix}; \begin{matrix}[1^2]\\ [0]_0\end{matrix}\right\rangle$	$+2^{1/2}\left[\frac{1}{n}\right]^{1/2}$
	$\left\langle \begin{matrix}[2]\\ [1]_{-1}\end{matrix} \middle\vert \begin{matrix}[1^2]\\ [0]_1\end{matrix}; \begin{matrix}[1^2]\\ [1]_{-1}\end{matrix}\right\rangle$	$+1/2^{1/2}\left[\frac{1}{n-2}\right]^{1/2}$

respectively. Since the little group SU(p)$\times$U(1)$\times$SU(q)$\times$U(1) has rank two less than Sp($2p+2q$), we must define two independent shifting operators, to specify the phases:

$$\Xi^{+-} = \Xi_{p\,2n}$$

$$\Xi^{--} = -\Xi_{p+n\,2n}.$$

Finally, note that in view of (18) and (21), there are now two independent SU(p)$\times$U(1)$\times$SU(q)$\times$U(1) singlets in $\langle 2\rangle$, occurring in the submultiplets $\langle 2\rangle\times\langle 0\rangle$ and $\langle 0\rangle\times\langle 2\rangle$ of Sp($2p$)$\times$Sp($2q$) (step 3.4).

In table 6, the tensors Q_{AB} (see (19)) and Ξ_{AB} are decomposed with respect to SU(p)$\times$U(1)$\times$SU(q)$\times$U(1) (only the common submultiplets are shown: cf (18) and (21)). States belonging to each submultiplet are shown, together with their normalisations N^2, and relative phases (up to an overall sign). The calculation proceeds as in the previous two examples; care must be exercised in the application of the phase conventions. The singlet factors thus calculated are summarised in table 8.

6. Conclusions

In this paper we have developed tensor operator techniques, appropriate to the investigation of assumptions about patterns of symmetry breaking, in the context of perturbation theory. We have retrieved familar results of the Gell-Mann Okubo type for expectation values, and with the same formalism, we have been able to evaluate transition matrix elements. The latter extension is particularly significant in the cases of groups O(n) and Sp(n), where mixing problems between degenerate multiplets are generally more severe (for example, between symmetric and antisymmetric parts, [2] and $[1^2]$ or $\langle 2\rangle$ and $\langle 1^2\rangle$), than in the case of SU(n) (for example, between the adjoint, $\{\bar{1};1\}$ and singlet $\{0\}$).

Table 8. Singlet factors of $SU(p)\times U(1)\times SU(q)\times U(1)$ for $\langle 2\rangle\times\langle 2\rangle$ contains $\langle 1^2\rangle$ of $Sp(2p+2q)$.

$\langle 1^2\rangle$ state	$\langle 2\rangle$ Breaking term $\left\lvert \begin{matrix}\langle 2\rangle\times\langle 0\rangle \\ \{0\}_0\times\{0\}_0\end{matrix}\right\rangle$	$\left\lvert \begin{matrix}\langle 0\rangle\times\langle 2\rangle \\ \{0\}_0\times\{0\}_0\end{matrix}\right\rangle$	$\langle 2\rangle$ state
$\left\langle \begin{matrix}\langle 1\rangle\times\langle 1\rangle \\ \{1\}_1\times\{1\}_1\end{matrix}\right\rvert$	$+\frac{1}{2}\left(\frac{1}{p(n+1)}\right)^{1/2}$	$+\frac{1}{2}\left(\frac{1}{q(n+1)}\right)^{1/2}$	$\left\lvert \begin{matrix}\langle 1\rangle\times\langle 1\rangle \\ \{1\}_1\times\{1\}_1\end{matrix}\right\rangle$
$\left\langle \begin{matrix}\langle 1\rangle\times\langle 1\rangle \\ \{1\}_1\times\{\bar{1}\}_{-1}\end{matrix}\right\rvert$	$+\frac{1}{2}\left(\frac{1}{p(n+1)}\right)^{1/2}$	$-\frac{1}{2}\left(\frac{1}{q(n+1)}\right)^{1/2}$	$\left\lvert \begin{matrix}\langle 1\rangle\times\langle 1\rangle \\ \{1\}_1\times\{\bar{1}\}_{-1}\end{matrix}\right\rangle$
$\left\langle \begin{matrix}\langle 1^2\rangle\times\langle 0\rangle \\ \{\bar{1}1\}_0\times\{0\}_0\end{matrix}\right\rvert$	$+\left(\frac{1}{p(n+1)}\right)^{1/2}$	0	$\left\lvert \begin{matrix}\langle 2\rangle\times\langle 0\rangle \\ \{\bar{1}1\}_0\times\{0\}_0\end{matrix}\right\rangle$
$\left\langle \begin{matrix}\langle 0\rangle\times\langle 0\rangle \\ \{0\}_0\times\{0\}_0\end{matrix}\right\rvert$	$+\frac{1}{2}\left(\frac{q}{pn(n+1)}\right)^{1/2}$	$+\frac{1}{2}\left(\frac{p}{qn(n+1)}\right)^{1/2}$	$\left\lvert \begin{matrix}\langle 0\rangle\times\langle 0\rangle \\ \{0\}_0\times\{0\}_0\end{matrix}\right\rangle$
$\left\langle \begin{matrix}\langle 0\rangle\times\langle 1^2\rangle \\ \{0\}_0\times\{\bar{1}1\}_0\end{matrix}\right\rvert$	0	$+\left(\frac{1}{q(n+1)}\right)^{1/2}$	$\left\lvert \begin{matrix}\langle 0\rangle\times\langle 2\rangle \\ \{0\}_0\times\{\bar{1}1\}_0\end{matrix}\right\rangle$
$\left\langle \begin{matrix}\langle 1\rangle\times\langle 1\rangle \\ \{\bar{1}\}_{-1}\times\{\bar{1}\}_{-1}\end{matrix}\right\rvert$	$+\frac{1}{2}\left(\frac{1}{p(n+1)}\right)^{1/2}$	$+\frac{1}{2}\left(\frac{1}{q(n+1)}\right)^{1/2}$	$\left\lvert \begin{matrix}\langle 1\rangle\times\langle 1\rangle \\ \{\bar{1}\}_{-1}\times\{\bar{1}\}_{-1}\end{matrix}\right\rangle$
$\left\langle \begin{matrix}\langle 1\rangle\times\langle 1\rangle \\ \{\bar{1}\}_{-1}\times\{1\}_1\end{matrix}\right\rvert$	$+\frac{1}{2}\left(\frac{1}{p(n+1)}\right)^{1/2}$	$-\frac{1}{2}\left(\frac{1}{q(n+1)}\right)^{1/2}$	$\left\lvert \begin{matrix}\langle 1\rangle\times\langle 1\rangle \\ \{\bar{1}\}_{-1}\times\{1\}_1\end{matrix}\right\rangle$

In the literature of the past few decades, the multitude of approaches taken to the solution of group-theoretical problems in physics fall into two broad categories (Baird and Biedenharn 1963). What may be termed 'global' methods (see for example Weyl 1939) include classical tensor and character techniques, having their origins in the work of the great algebraists at the turn of this century. The comparatively more recently developed 'local' or 'algebraic' methods, including formal boson operator methods, realisations on spaces of homogeneous functions, and so on, were spurred on by developments in physics, especially in spectroscopy (see for example Wigner 1959, Racah 1965). Most recently, however, physics has turned again to embrace the global approach as well. The techniques advocated in this paper can be viewed as intermediate in nature, in that both tensor and Lie algebraic methods are exploited to good effect. This situation is possible partly because the formulation of the problem, entailing as its does a very special form for the symmetry breaking term as a singlet state with respect to the little group, removes the necessity for much of the Young symmetrisation which normally plagues the classical tensor methods.

It is well to emphasise here some of the limitations of our techniques. One such has been pointed out already: we have been exclusively concerned with the realisation of irreducible tensor operators in the enveloping algebra, which corresponds to irreducible constituents of symmetrised Kronecker powers of the adjoint representation. In order to generalise our work further, we would need to introduce, for example, a tensor operator transforming as the defining representation (playing the role of a 'boson' operator). Also, in our examples we have not gone further than symmetrised quadratic order in the generators. Our method, also generalised in the above sense, still applies to higher-order constructs; the only difficulty lies in writing down the eigenvalues of a generating set of the higher order little group invariants. However, recent work by Green (1975), Rashid and Nwachuku (1978), and Edwards (1978) considerably simplifies this task. Work on problems of higher-order examples is continuing.

Our examples (§§ 4 and 5) only involved little sub-algebras which were regular sub-algebras of the classical simple Lie algebras. The method obviously applies equally to non-regular sub-algebras. For example, the non-simple maximal S-sub-algebras (Dynkin 1952, Lorente and Gruber 1972) $\mathrm{SU}(3)\times\mathrm{SU}(2)\subset\mathrm{SU}(6)$, $\mathrm{SU}(2)\times\mathrm{SU}(2)\subset\mathrm{SU}(4)$ are of great physical relevance. It is known that in the former case, expectation values of symmetry breaking terms involve two 'non-commuting labelling chains' (see for example Pais 1966). It is likely that similar complications obtain for other non-regular embeddings, although in many cases, such as the maximal simple S-sub-algebra $\mathrm{SU}(3)\subset\mathrm{SO}(8)$, other measures, such as non-sub-group labels, are called for (see also the list of suggested applications given below).

It is of interest to establish the general validity of the enveloping algebra realisations for all admissible irreducible tensor operators, on the same footing as it is known for the adjoint representation (Okubo 1975, Rashid and Nwachuku 1976). As was explained in § 3, in our work we have made this assumption throughout, but have checked it case by case. Further work in this direction is in progress.

It might be thought that the calculation in § 5, of singlet factors of the sub-group K, for $\{\alpha\}$ contained in $\{\beta\}\times\{\gamma\}$ of K, where the K-submultiplet is the singlet $\{0\}_0$, admits of a more general solution, for example in terms of the eigenvalues of a generating set of invariants of $\{\alpha\}$ and $\{\gamma\}$ and of the appropriate K-submultiplet $\{\kappa\}_X$ (in analogy with the algebraic formulae of § 4). Such a solution would be applicable to general $\{\alpha\}$ and $\{\gamma\}$. However, in the course of this investigation, such a solution has not been found.

Finally, having seen its usefulness in symmetry breaking, we mention briefly some other applications of this work. In its own right, it provides a powerful method of evaluating Chebsch-Gordan coefficients. It differs from what might be called 'quark state' techniques in that only the required information is extracted, whereas in the latter method, each irreducible multiplet must be fully constructed explicitly in terms of basic states. Lastly, we mentioned above the possible difficulties with subgroup chains involving a state labelling problem; in fact, our realisations may be useful in its resolution. For example, it is hoped to be able to apply this to the $\mathrm{O}(n)\subset\mathrm{SU}(n)$ problem (Jarvis 1974, Green *et al* 1976).

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